Compact covers and ℓ-equivalence
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For a Tychonoff space X, we denote by $C_p(X)$ and $C_c(X)$ the space of continuous real-valued functions on X equipped with the topology of pointwise convergence and the compact-open topology, respectively.

Recall that Nagata's theorem states that if the topological rings $C_p(X)$ and $C_p(Y)$ are topologically isomorphic, then X and Y are homeomorphic. From this result follows that two spaces X and Y are said to be *t-equivalent* (ℓ_p*-equivalent*) [ℓ_c*-equivalent*] if the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic (linearly homeomorphic) [$C_c(X)$ and $C_c(Y)$ are linearly homeomorphic]. Also we say that a topological property $\mathcal{P}$ is preserved by t-equivalence (ℓ_p-equivalence) [ℓ_c-equivalence] if whenever two spaces X and Y are t-equivalent (ℓ_p-equivalent) [ℓ_c-equivalent] and X has the property $\mathcal{P}$, Y has the property $\mathcal{P}$ too.

The natural question asking *What topological properties are preserved by the relations of t-equivalence, ℓ_p-equivalence or ℓ_c-equivalence?* comes from Arkhangel'skii problem asking if *a first countable space Y which is ℓ_p-equivalent to a metrizable space X must also be metrizable.* J. Baars, J. de Groot and J. Pelant proved that *complete metrizability is preserved by ℓ_p-equivalence in the class of metrizable spaces.* They also gave an alternative proof for separable metrizable spaces by using Christensen's Theorem (A metrizable space X is Polish if and only if X admits a compact resolution swallowing compact sets). Therefore: *The property of being a completely metrizable space is preserved by the ℓ_p-equivalence for spaces satisfying the first axiom of countability.* Later on, Valov proved that the answer to the Arhangel'skii's problem is positive for Čech-complete spaces Y.

We will present the preservation of complete metrizability by ℓ_c-equivalence. Our proof of the preservation of complete metrizability in the case of ℓ_c-equivalent spaces X and Y, where X is Polish and Y is first-countable follows from the existence of a quotient linear map from $C_c(X)$ onto $C_c(Y)$. The non-separable case is an open problem. Our approach uses a property of $C_c(X)$ which is preserved by linear open maps and characterizes spaces X with a compact resolution swallowing compact sets.

Other properties preserved by ℓ_c-equivalent will be considered and also other questions related with ℓ_p-equivalence will be presented. For instance, a new characterization of the Lindelöf Σ-property of X in terms of $C_p(X)$, enables extend some Okunev's results by showing that if there exists a surjection from $C_p(X)$ onto $C_p(Y)$ (respectively from $L_p(X)$ onto $L_p(Y)$) that takes bounded sequences to bounded sequences, then υY is a Lindelöf Σ-space (respectively K-analytic) if υX has this property.

Basic references

1. J. Kakol, W. Kubis and M. López-Pellicer, *Descriptive topology in Selected Topics of Functional Analysis*, Developments in Mathematics **24**, Springer, New York, Dordrecht, Heidelberg, London, 2011.

2. J. Kakol, M. López-Pellicer, O. Okunev, *Compact cover and Function Spaces*, submitted.

3. O. Okunev, *A relation between spaces implied by t-equivalence*, Topology Appl. **158** (2011) 2158-2164.